Mathematics Underlying the Physics First Curriculum

Implications for 8th and 9th Grade Mathematics

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An ongoing collaboration between math and physics teachers is an integral part of a successful implementation of the Physics First curriculum. This summary is intended as the springboard for such a collaboration. The list below contains some examples from the Physics First curriculum which you, the mathematics teacher, may incorporate into your lessons and/or homework assignments for 8th and 9th grade.

1. Notation

Students need to understand that in a physics course the quantities measured in an experiment dictate the notation used for the independent and dependent variables.

- **the letters x and y are NOT the letters generally used to label the horizontal and vertical axes in a physics class**
  While in most mathematics textbooks the letter x is used for the independent variable (represented on the horizontal axis) and the letter y is used for the dependent variable (represented on the vertical axis), this is not typically the case in the Physics First course. For example, in the study of uniform motion, time is an independent variable, denoted by t and represented on the horizontal axis, while position is a dependent variable, denoted by x and represented on the vertical axis. When sketching graphs, use often the terms **independent** variable/**dependent** variable and emphasize the fact that the actual symbols used for these variables do not affect the shape of the graph.

  Word of caution: when using examples from Physics First in your mathematics class, make sure the terminology is the one students use in the physics class. For example, when discussing the motion of an object along a straight line, there is a clear distinction between *distance traveled* and *change in position* and between *speed* and *velocity*. The distance traveled is the total length traveled and is always positive, while change in position is the difference between the final position and the initial position, which may be negative. Speed is always positive while velocity may be positive or negative, the sign indicates the direction in which the motion takes place.
- **the subscripts used in Physics First are not always numerical**
  For example, v_f and v_i denote the final velocity and the initial velocity, respectively, as opposed to the v_2 and v_1 option that would be the preferred choice in a mathematics textbook.
- **there is a considerable use of the symbol Δ in Physics First to denote change**
  For example, if x_i denotes the initial (or starting) position and x_f denotes the final (or ending) position, then $\Delta x = x_f - x_i$ denotes the change in position. Similarly, the time change from t_i to t_f will be denoted by $\Delta t = t_f - t_i$.

2. Literal Formulas

- **Use examples requiring students to substitute values into a given formula.**
  Example: Include examples with variables denoted by other letters than just x and y and whenever appropriate include formulas from the Physics First curriculum, such as: Evaluate $3ma$ if m = 4 and a = 7.5. More examples

3. Literal equations

- **Use examples requiring students to solve for one variable in terms of the others.** A good resource is the list of formulas studied in the Physics First class. (see attached at the end of this document)

4. Linear functions

- **slope as a rate of change of the dependent variable with respect to the independent variable**
  Emphasize the computation and interpretation of the slope of a line as a rate of change of the dependent variable with respect to the independent variable, specifying units whenever possible. Include examples where the independent and dependent variables are denoted by other letters than x and y.
- **changing the scale on the dependent/independent axis manipulates the graph's appearance**
  Give a few examples to show that changing the scale on the dependent axis manipulates the graph's appearance, making the slope of the graph of a linear function appear to be greater or lesser than before the manipulation. As such, if the slopes of the graphs of two linear functions are to be compared only by looking at their graphs, the scales used for the two functions should be the same. This also shows the importance of using units of measure for the variables involved. Examples are shown in Figs. 1 and 2 below, where the *values of the slopes in both figures are the same*, even though the line in Fig. 1 looks steeper because of the scale of the vertical axis.

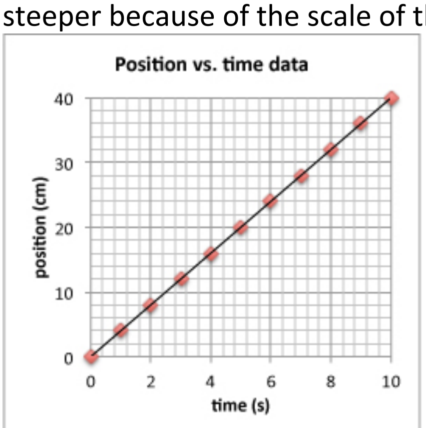


Fig. 1

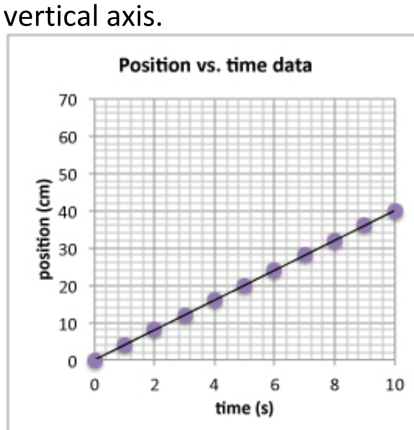


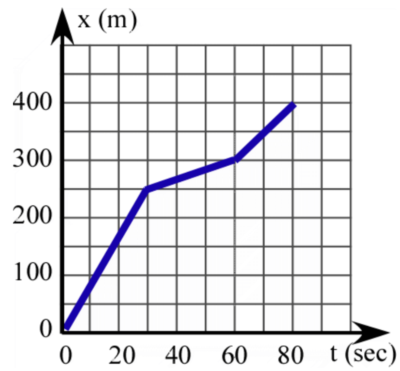
Fig. 2

- **linear functions= functions for which the rate of change over any interval is a constant**
  Stress the fact that linear functions are precisely those functions for which the rate of change over any interval is a constant which does not change from one interval to

another. Help students make the connection between constant ratios and linear relationships.

- **uniform motion**
  Uniform motion problems typically involve something traveling at some fixed and steady (uniform) pace (rate or speed) and the main governing formula is $d = st$, where d stands for distance (position), s stands for the (constant or average) speed, and t stands for time. As such, examples in the spirit of the ones in the Uniform Motion unit are well suited when discussing linear functions.

5. Piecewise linear functions



- Expose students to examples of motion (of a car, of a bicycle, etc.) depicted by piecewise linear graphs ("broken" lines). Use t (time) as the independent variable and x (position) as the dependent variables, each with appropriate units of measure. (More examples)

6. Quadratic functions

- **provide students with an introduction to quadratic functions and their graphs (parabolas) as a first-semester topic in Algebra I**
  Provide students with an introduction to quadratic functions and their graphs (parabolas) as a first-semester topic in Algebra I without solving quadratic equations based on the quadratic formula or factoring (this is a topic that will likely be covered during the second semester; see the problems listed below to be used after you discuss solving quadratic equations).
- **include evaluating quadratic expressions**
  For example, have them evaluate $y = 5x^2$ for $x = 0$, $x = 2$, $x = 3$ and graph parabolas using calculators.
- **include the uniform accelerated motion under gravity (free fall)**
  After covering the quadratic formula and factoring, you can use problems related to uniform accelerated motions under gravity (free fall). (More examples)

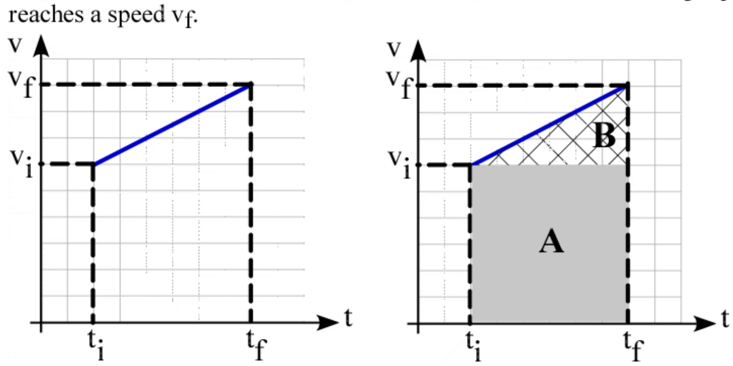
7. Solving systems of equations

- **introduce systems of equations as a second-semester topic**
  In addition to solving the equation $2x-6 = 4-x$ (sometimes referred to as "solving simultaneous quations") using symbol manipulation, consider having students graph the lines $y_1 = 2x - 6$ and $y_2 = 4 - x$ and decide if the two lines intersect. Emphasize the connection between the solution for the original equation and the x coordinate of the intersection point of the two lines.
- **use the trace feature of a graphing calculator to estimate the point of intersection of two given lines**
  Explore finding approximate solutions to systems of equations using the trace feature of graphing calculator to estimate the point of intersection of two given lines. Use the intersect function if available on your calculator to compute exact solutions.

8. Area

- **area of polygonal shapes**
  When determining the area of geometric figures, include examples of polygonal shapes for which the students do not have a direct area formula but which can be partition into familiar shapes (e.g., triangles, rectangles), find the areas of each partition, and sum the areas to get the total area.
- **area of the region under the graph of a piecewise linear function**
  Include a few in-class or homework problems in which students have to compute the area of the region under the graph of a piecewise linear function.

Object starts at an initial speed v_i, accelerates, and after time $\Delta t = t_f - t_i$, reaches a speed v_f.



An example is shown in the figure. The area under a velocity-time graph gives the displacement of the object. Thus the displacement in the time interval between t_i and t_f is given by the area under the blue line. That area is the sum of the area of the rectangle A and the triangle B.

9. Measurement

- **the metric system**
  Introduce elements of the metric system early in the school year, especially units of length (e.g., km, m, cm, mm).
- **unit conversion**
  Provide several examples of unit conversions. Include problems in which the data given is not all in the same units of measure (e.g., if the speed of a cyclist is 2 meters per second, how long will it take her to cover 1 kilometer?). Require students to include the unit of measure in every step when solving a problem that involves quantities given in units of measure.

10. Number and Operation

- **include decimal numbers in examples and problems**
  For example, include the computation of the quotient of two decimal numbers in computing the slope of a line or evaluating algebraic expressions for decimal values.
- **Encourage proportional reasoning by including appropriate problems as you cover various topics.**

Sample problems

A list of suggested problems is included below. This is just a small list of sample problems which you can expand as appropriate. Keep in mind that this might be done often by just rephrasing a typical problem from your mathematics textbook. For example, a problem stated as:

Solve for x in the equation $2x - 4 = 5 + 3x$

may be rephrased as:

Solve for Δt in the equation $2\Delta t - 4 = 5 + 3\Delta t$

While this type of "adjustment" is not expected to be done for many mathematics problem, including some examples of this type will not only help students' understanding in their physics class, but also in their understanding of the mathematical concepts, as this should reinforce the principle that the notation used is immaterial for the problem at hand.

1. Evaluate $3ma$ if $m = 4$ and $a = 7.5$.

2. If $R = \rho\frac{L}{A}$, evaluate R when $\rho = 10^{-4}$, $L = 12$, $A = 9\pi$.

3. Evaluate $v \cdot t + x_0$ if $v = 3\,\frac{m}{s}$, $t = 3s$ and $x_0 = 2.3\,m$.

4. Evaluate $\rho^4 + 2ac$ if $\rho = -2$, $a = -3$, $c = 10$.

5. Evaluate $v_i + at$ if $v_i = 2\frac{m}{s}$, $a = -4\frac{m}{s^2}$ and $t = 1s$.

6. If $v = \dfrac{x}{t}$ evaluate x, when $v = 120\frac{m}{s}$ and $t = 2\,min$.

7. If $F = ma$, evaluate m when $F = 10\frac{kg\cdot m}{s^2}$ and $a = 2\frac{m}{s^2}$.

8. If $v_s = v_i + 2a \cdot \Delta y$, solve for Δy provided $a \neq 0$.

9. Solve for Δx in the equation $2(\Delta x) + 4 = 3\big(5(\Delta x) - 6\big)$.

10. Solve for v_f in the equation $3v_f - 2 = 4v_s + 9$.

11. Solve for Δx in the equation $\dfrac{\Delta x}{8-4} = 12\Delta x + 3$.

12. Find the slope of the line that passes through the points $(6, 1)$ and $(-2, 5)$ by using the formula slope $= \dfrac{\Delta y}{\Delta x}$, where $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$.

13. Solve for x if $\dfrac{3.75}{x} = \dfrac{32}{7.3}$

14. Evaluate m if $m = \dfrac{z_f - z_i}{t_f - t_i}$ if $z_f = 13.7$, $t_f = 15.8$, $z_i = 3.625$, and $t_i = 5.32$.

15. Simplify the given expressions. Make sure that you respect the order of operations.

$$\frac{3(0.2-5)+0.3}{2^2(3+4.2^2)} \qquad\qquad 8 + 2 \times 5 \times 34 : 9$$

16. Evaluate $y_i + v_0 \cdot \Delta t + \frac{1}{2}a(\Delta t)^2$ if $y_i = 2m$, $v_0 = 3m/s$, $a = -9.8m/s^2$ and $\Delta t = 0.3s$.

17. Given that $V = IR$ and $R \neq 0$, solve for I. What is the value of I if $V = 7$ and $R = 5$?

18. $d = v \cdot t$, solve for v if $t \neq 0$.

19. If $v_f = v_i + a \cdot \Delta t$, solve for a knowing that $\Delta t \neq 0$. What is the value of a when $v_f = 10m/s$, $v_i = 34m/s$ and $\Delta t = 6s$?

20. The resistance of a cylindrical wire measured in Ohms (the notation for Ohm is Ω) is given by the formula $R = \rho \frac{L}{A}$, where ρ is the resistivity of the material used to make the wire measured in Ohms times cm (write $\Omega\,\mathrm{cm}$), L is the length of the wire measured in cm, and A is the area of the cross section of the wire measured in $(\mathrm{cm})^2$.

• The resistivity of silver is $1.59 \times 10^{-6}\ \Omega\,\mathrm{cm}$ and that of gold is $2.44 \times 10^{-6}\ \Omega\,\mathrm{cm}$. Compute the resistance of a cylindrical wire of length 10 cm, of diameter 2 cm and made out of silver. What will the resistance of a wire of the same dimensions be if it is made of gold?

• What is the radius of a cylindrical wire made of nichrome which has a length of 100π cm and resistance equal to $0.25\ \Omega$? We know that the resistivity of nichrome is $10^{-4}\ \Omega$ cm.

21. The total resistance R of two resistors R_1 and R_2 connected in parallel is given by the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If the total resistance of two light bulbs connected in parallel is $R = \frac{15}{19}\,\Omega$ and if one has resistance $R_1 = \frac{3}{2}\,\Omega$, determine the resistance of the second light bulb.

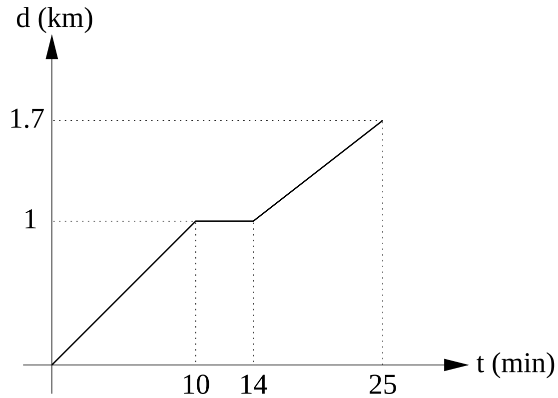
22. If $y_f = \frac{1}{2}a(\Delta t)^2$, solve for Δt knowing that $a \neq 0$.

23. If $h \neq 0$, solve for b_1 in the equation $\frac{1}{2}h(b_1 + b_2) = A$.

24. Emily and Michell are driving their tractors in opposite directions. Emily is driving at 34 miles/hour while Michell is driving at 26 miles/hour. How many hours will it be before they are 30 miles apart?

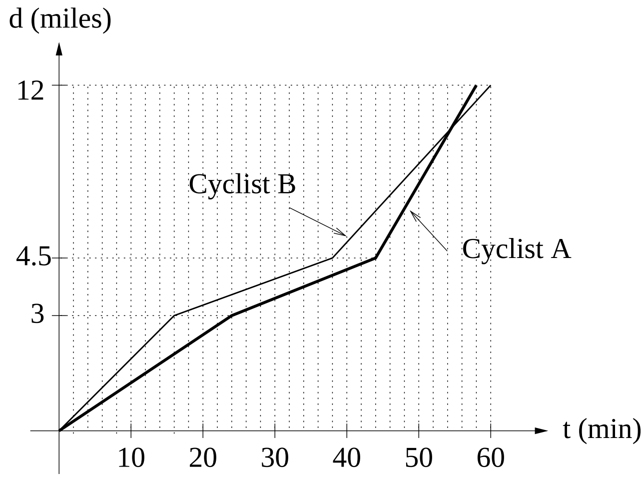
25. A car traveling at 60 miles/hour overtakes a cyclist who, has a speed of 15 miles/hour and had a 4 hour head start. How far from the starting point does the car overtake the cyclist?

26. Mary walks to school each morning. On her way, she goes by Emily's house and then together they finish the walk to school. Here is the graph representing the distance traveled by Mary during her walk to school.



(a) How long it takes Mary to get from home to school?
(b) For how long does Mary wait for Emily?
(c) What is the distance that Mary travels from home to school each morning?

27. The graphs of the distances covered by two cyclists, A and B, in a race is sketched below.



(a) Who won the race and in how much time?

(b) There is a very steep hill in this course. How many miles into the race do the riders hit the hill? How do you know?

(c) Which competitor had the fastest start? How do you know?

28. John is going from home to his grandparents' house. This trip takes 10 minutes. For the first 5 minutes he travels a distance of 500 meters at a constant speed. He slows down for the next 2 minutes, still walking at a constant speed, but traveling only an additional 100 meters. The rest of the route he decides to run at a constant speed that is twice the speed he had for the first 5 minutes of his trip.

(a) What is the speed at which John was running?

(b) Determine John's speed during the following periods of his trip: for the first 3 minutes, from minute 5 to minute 6, for the last 2 minutes.

(c) Graph the distance traveled by John as a function of time. When labeling your axes specify the units of measure.

(d) How long would have taken John to get from home to his grandparents' house if he were to run at the speed he had during the last 100 meters of his trip?

28. Use the finite difference method for polynomial functions to contrast between linear functions and quadratic functions.

(i) For linear functions (uniform motion) the first differences are all equal to a constant. If the position x is a linear function of time t, $x = at + b$, then the first differences are equal to a. The table below corresponds to the linear function $x = 5t$.

t	1		2		3		4		5
x	5		10		15		20		25
first differences		5		5		5		5	

(ii) For quadratic functions (accelerated motion) the second differences are all equal to a constant. If the position x is a quadratic function of time t, $x = at^2 + bt + c$, then the first differences are not equal but the second differences are equal to $2a$. The table below corresponds to $x = 3t^2$.

t	1		2		3		4		5
x	3		12		27		48		75
first differences		9		15		21		27	
second differences			6		6		6		

29. An object is launched at 19.6 meters per second (m/s) from a 58.8-meter tall platform. The equation for the object's height h at time t seconds after launch is $h(t) = -4.9t^2 + 19.6t + 58.8$, where h is in meters. When does the object strike the ground?

30. Adrian received a model rocket for his birthday. The launcher is powered by an air pump, and the height attained by the rocket depends on the number of times he pumps air. Adrian knows that 5 pumps will project the rocket upward according to the formula $h(t) = -16t^2 + 128t$, where $h(t)$ is the height of the rocket, measured in feet from the ground, t seconds after take off.

(a) Sketch the graph of $h(t)$ for $t \geq 0$. Is this the trajectory of the rocket? Explain.

(b) After how many seconds will the rocket start falling? What will be the maximum height attained by the rocket?

(c) After how many seconds will the rocket hit the ground?

31. Your physics teacher keeps telling you a car is traveling at 22.35 m/s. How fast would the speedometer in the car indicate it going in miles/hour?

32. You are traveling at a constant speed of 55 miles/hour over a bridge that is 4260 feet long. How long does it take to cross the bridge?

33. If $x = 23.6$ miles and $t = 4.2$ hours, evaluate and simplify the expression bellow. Your answer should be in meters/seconds:

$$\frac{x}{t} \cdot \frac{1 \text{ miles}}{1609.344 \text{ meters}} \cdot \frac{1 \text{ hour}}{3600 \text{ seconds}}$$

34. How tall are you to the nearest centimeter?

35. During 2008, a car company registers the following profits: 1 million dollars in January, 1.4 million dollars in February, 1.5 million dollars in March, 12.0 million dollars in each of the months of April, May, and June, and 2.4 million dollars in July. What was the average daily profit of this company for the period January-July.

36. Sue can run at the rate of 7000 inches per minute. Bob can run at a rate of 120 centimeters per second. If Bob and Sue run in a race of 100 meters, who will be the winner? Suppose Sue and Bob run at the given rates for 44 seconds and the distances they cover are denoted by d_S and d_B, respectively. Compute d_S and d_B.

37. Assume you are a superhero that runs at a superhuman speed of 20,000,000 feet/hour. How many miles/hour do you run? What is your running speed in miles/minute?

38. Amy traveled 9 miles in the first hour. This is 1/8 times more than how long she traveled during the second hour, and 1/5 times more than she traveled during the third hour. What is the total distance she covered in these three hours?

Here are a few examples of word problems related to uniform motion which you can use when teaching students analytic methods of solving systems of equations (e.g., by substitution, elimination).

39. A 300-mile, 3-hour plane trip was flown at two speeds. For the first part of the trip, the average speed was 95 miles per hour. Then the tailwind picked up, and the second part of the trip was flown at an average speed of 105 miles per hour. For how long did the plane fly at each speed?

40. A business man had to attend a meeting. He drove from home at an average speed of 45 miles per hour to an airport where he boarded a plane. The plane flew him to the meeting place at an average speed of 85 miles per hour. The entire distance was 260 miles and the entire trip lasted four hours. Find the distance from home to the airport.

41. A car leaves a buss station 1 hour after a bus left from the same station. The bus is traveling 15 miles per hour slower than the car. Find the average speed of the car and that of the bus, if the car overtakes the bus in four hours.

A partial list of formulas used in Physics First

$$R = \frac{\rho L}{A}$$

$$V = IR$$

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$P = VI$$

$$E = P\,\Delta t$$

$$\Delta x = x_f - x_i$$

$$\Delta t = t_f - t_i$$

$$d = |x_f - x_i|$$

$$x_f = v(\Delta t) + x_i$$

$$v = \frac{\Delta x}{\Delta t}$$

$$v_f = v_i + a\Delta t$$

$$v_f^2 - v_i^2 = 2a\Delta x$$

$$F = mg$$

$$y_f = y_0 + v_0(\Delta t) + \frac{1}{2}a(\Delta t)^2$$